UNIVERSITY OF COPENHAGEN
DEPARTMENT OF ECONOMICS

2nd year of study. 2012 W-2DM ex sol

WRITTEN EXAM. DYNAMIC MODELS

Wednesday, January 18, 2012

SOLUTIONS

Problem 1. We consider the polynomial $P : \mathbf{C} \to \mathbf{C}$ that is given by

$$\forall\, z \in \mathbf{C} : P(z) = 2z^4 + 2z^3 + 7z^2 + 2z + 5.$$

Furthermore, we consider the differential equation

$$(*) \qquad \frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + \frac{7}{2}\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{5}{2}x = 0$$

and the differential equations

$$(**) \qquad \frac{d^4x}{dt^4} + \frac{d^3x}{dt^3} + \frac{7}{2}\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{5}{2}x = 27e^t$$

and

$$(***) \qquad \frac{d^5y}{dt^5} + \frac{d^4y}{dt^4} + \frac{7}{2}\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{5}{2}\frac{dy}{dt} = 0.$$

(1) Show that the complex numbers i and $-i$ are roots of the polynomial P, i. e. $P(i) = 0$ and $P(-i) = 0$.

Solution. We easily find that $P(i) = P(-i) = 0$. Hence i and $-i$ are roots of the complex polynomial P.

(2) Solve the equation

$$P(z) = 0.$$

Solution. Using polynomial division we obtain that

$$\forall\, z \in \mathbf{C} : P(z) = (z - i)(z + i)(2z^2 + 2z + 5),$$

and furthermore we find that

$$2z^2 + 2z + 5 = 0 \Leftrightarrow z = \frac{-2 \pm \sqrt{4 - 40}}{4} \Leftrightarrow z = -\frac{1}{2} \pm \frac{3}{2}i.$$

Now, we have shown that the polynomial P has the roots:

$$z_1 = i,\ z_2 = -i,\ z_3 = -\frac{1}{2} + \frac{3}{2}i, \text{ and } z_4 = -\frac{1}{2} - \frac{3}{2}i.$$

(3) Determine the general solution of the differential equation $(*)$.

Solution. The characteristic polynomial P_C of the differential equation $(*)$ is $P_C = \frac{1}{2}P$, and the characteristic roots are exactly the roots of the polynomial P. This implies that the general solution of $(*)$ is

$$x = c_1 \cos(t) + c_2 \sin(t) + c_3 e^{-\frac{1}{2}t} \cos(\frac{3}{2}t) + c_4 e^{-\frac{1}{2}t} \sin(\frac{4}{2}t),$$

where $c_1, c_2, c_3, c_4 \in \mathbf{R}$.

(4) Show that the differential equation $(*)$ is not globally asymptotically stable.

Solution. Since the functions cos and sin don't have any limit as t is approaching to infinity the differential equation $(*)$ is not globally asymptotically stable.

(5) Determine the general solution of the differential equation $(**)$.

Solution. Since the function $t \to e^t$ is not a solution of the homogeneous differential equation $(*)$ we know that the function $\hat{x} = Ae^t$ will be a solution of the inhomogeneous differential equation $(**)$ for some value of the constant A.

We have that

$$\hat{x}' = \hat{x}'' = \hat{x}''' = \hat{x}'''' = Ae^t,$$

and then we find that $A = 3$.

The general solution of the differential equation is

$$x = c_1 \cos(t) + c_2 \sin(t) + c_3 e^{-\frac{1}{2}t} \cos(\frac{3}{2}t) + c_4 e^{-\frac{1}{2}t} \sin(\frac{3}{2}t) + 3e^t,$$

where $c_1, c_2, c_3, c_4 \in \mathbf{R}$.

(6) Determine the general solution of the differential equation $(***)$.

Solution. The characteristic polynomial $Q : \mathbf{C} \to \mathbf{C}$ of the differential equation $(***)$ is given by

$$\forall\, z \in \mathbf{C} : Q(z) = z^5 + z^4 + \frac{7}{2}z^3 + z^2 + \frac{5}{2}z = z\Big(z^4 + z^3 + \frac{7}{2}z^2 + z + \frac{5}{2}\Big).$$

We notice that the roots of Q are

$$z_0 = 0,\ z_1 = i,\ z_2 = -i,\ z_3 = -\frac{1}{2} + \frac{3}{2}i, \text{ and } z_4 = -\frac{1}{2} - \frac{3}{2}i.$$

From this we find that the general solution of $(***)$ is

$$y = c_0 + c_1\cos(t) + c_2\sin(t) + c_3e^{-\frac{1}{2}t}\cos(\frac{3}{2}t) + c_4e^{-\frac{1}{2}t}\sin(\frac{3}{2}t),$$

where $c_0, c_1, c_2, c_3, c_4 \in \mathbf{R}$.

Problem 2. Consider the vector space $\mathbf{R}^n$, where $n \in \mathbf{N}$ and $n \geq 3$. Also consider the set

$$S = \{x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : x_1 > 0 \wedge x_2 > 0\}.$$

(1) Show that the set S is an open subset of $\mathbf{R}^n$.

Solution. Let the point $a = (a_1, a_2, \ldots, a_n) \in S$ be arbitrarily chosen. Then $a_1 > 0$ and $a_2 > 0$. Choose $r > 0$, such that

$$r \leq \min(a_1, a_2).$$

Then we consider the open ball

$$B(a, r) = \{x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n :$$

$$\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \ldots (x_n - a_n)^2} < r\},$$

and we notice that if $x \in B(a, r)$ then

$$|x_1 - a_1| < r \wedge |x_2 - a_2| < r \Leftrightarrow$$

$$a_1 - r < x_1 < a_1 + r \wedge a_2 - r < x_2 < a_2 + r.$$

Now we know that $B(a, r) \subset S$, and then we have shown that S is open.

(2) Find the closure $\overline{S}$ of the set S.

Solution. We claim that

$$\overline{S} = \{x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n : x_1 \geq 0 \land x_2 \geq 0\}.$$

Let us consider a point $a = (0, a_2, \dots, a_n)$ where $a_2 \geq 0$. Let $r > 0$ be any positive number. Then we consider the open ball

$$B_1(a, r) = \{x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n :$$

$$\sqrt{x_1^2 + (x_2 - a_2)^2 + \dots (x_n - a_n)^2} < r\}.$$

Now we find that if $x = (x_1, a_2, \dots, a_n) \in B_1(a, r)$ then

$$x_1^2 < r^2 \Leftrightarrow -r < x_1 < r,$$

and hence $a \in \partial S$. From this fact we easily verify the assertion.

(3) Find the complement $\mathcal{C}S$ of the set S and find the boundary $\partial(\mathcal{C}S)$ of this set.

Solution. We see that

$$\mathcal{C}S = \{x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n : x_1 \leq 0 \lor x_2 \leq 0\},$$

and then we notice that

$$\partial(\mathcal{C}S) = \{x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n :$$

$$(x_1 = 0 \land x_2 \geq 0) \lor (x_1 \geq 0 \land x_2 = 0)\},$$

(4) Is the set $\mathcal{C}S$ closed?

Solution. Since the set S is open the complement $\mathcal{C}S$ is closed.

Problem 3. We consider the vector valued function $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$\forall (x, y) \in \mathbf{R}^2 : f(x, y) = \begin{pmatrix} 2xy + e^y \\ e^x + 4y^2 \end{pmatrix}.$$

(1) Find the Jacobi matrix $Df(x, y)$ of the function f at any point $(x, y) \in \mathbf{R}^2$.

Solution. We find that

$$Df(x, y) = \begin{pmatrix} 2y & 2x + e^y \\ e^x & 8y \end{pmatrix}.$$

(2) Find the determinant $\det Df(x, y)$ and show that the Jacobi matrix $Df(0, 0)$ is non-singular.

Solution. We see that $\det Df(x, y) = 16y^2 - e^{x+y} - 2xe^x$ and that $\det Df(0, 0) = -1$. This shows that the Jacobi matrix

$$Df(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is non-singular.

(3) Prove that there exists a neighbourhood $U_{(0,0)}$ of the point $(0, 0)$ such that the Jacobi matrix $Df(x, y)$ is non-singular at any point $(x, y) \in U_{(0,0)}$.

Solution. Since all four entries of the Jacobi matrix $Df(x, y)$ are continuous functions, and since $Df(0, 0)$ is non-singular, the assertion is true.

(4) Find the inverse $(Df(0, 0))^{-1}$ of the non-singular Jacobi matrix $Df(0, 0)$.

Solution. We find that

$$(Df(0, 0))^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(5) Solve the equation

$$\begin{pmatrix} u \\ v \end{pmatrix} = f(0, 0) + Df(0, 0) \begin{pmatrix} x \\ y \end{pmatrix}$$

with respect to

$$\begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution. We find that

$$\begin{pmatrix} u \\ v \end{pmatrix} = f(0,0) + Df(0,0)\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} u-1 \\ v-1 \end{pmatrix} = Df(0,0)\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u-1 \\ v-1 \end{pmatrix} \Leftrightarrow x = v-1 \wedge y = u-1.$$

(6) Show that the vector valued function $g : \mathbf{R}^2 \to \mathbf{R}^2$ given by the rule

$$\forall (x,y) \in \mathbf{R}^2 : g(x,y) = f(0,0) + Df(0,0)\begin{pmatrix} x \\ y \end{pmatrix}$$

has no fixed points.

Solution. If the point (x,y) were a fixed point of the function g we would find that

$$x = y-1 \wedge y = x-1 \Rightarrow x = x-2 \Leftrightarrow 0 = -2.$$

Problem 4. We consider and the function $F : \mathbf{R}^3 \to \mathbf{R}$ given by the rule

$$\forall (t,x,y) \in \mathbf{R}^3 : F(t,x,y) = y^2 + (1+t^2)x.$$

Furthermore, we consider the functional

$$I(x) = \int_0^1 \Big(\Big(\frac{dx}{dt}\Big)^2 + (1+t^2)x\Big)\,dt.$$

(1) Show that for every $t \in \mathbf{R}$ the function $F = F(t,x,y)$ is convex in $(x,y) \in \mathbf{R}^2$.

Solution. We find that

$$\frac{\partial F}{\partial x} = 1+t^2 \text{ and } \frac{\partial F}{\partial y} = 2y,$$

and that the Hessian matrix of the function F is

$$F" = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

This matrix is positive semidefinite and hence the function F is a convex function of $(x,y) \in \mathbf{R}^2$.

(2) Solve the variational problem: Determine the minimum function $x^* = x^*(t)$ of the functional $I(x)$ subject to the conditions

$$x^*(0) = 3 \text{ and } x^*(1) = \frac{1}{24}\,.$$

Solution. Since the function F is a convex function of $(x, y) \in \mathbf{R}^2$ we know that the given variational problem is a minimum problem.

The Euler differential equation is:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\Big(\frac{\partial F}{\partial \dot{x}}\Big) = 0 \Leftrightarrow 1 + t^2 - 2\frac{d^2x}{dt^2} = 0 \Leftrightarrow$$

$$\frac{d^2x}{dt^2} = \frac{1}{2} + \frac{1}{2}t^2.$$

Now we find that

$$\frac{dx}{dt} = \frac{1}{2}t + \frac{1}{6}t^3 + c_1$$

and that

$$x = \frac{1}{4}t^2 + \frac{1}{24}t^4 + c_1 t + c_2,$$

where $c_1, c_2 \in \mathbf{R}$.

From the two given conditions

$$x^*(0) = 3 \text{ and } x^*(1) = \frac{1}{24}$$

we find that $c_1 = -\frac{13}{4}$ and that $c_2 = 3$.

Then we have that

$$x^* = x^*(t) = \frac{1}{4}t^2 + \frac{1}{24}t^4 - \frac{13}{4}t + 3.$$